Lesson 08 – Linear Programming

A mathematical approach to determine optimal (maximum or minimum) solutions to problems which involve restrictions on the variables involved.

Linear Programming Applications

Linear programming (LP) has been used to:

- establish locations for emergency equipment and personnel that minimize response time
- determine optimal schedules for planes
- develop financial plans
- determine optimal diet plans and animal feed mixes
- determine the best set of worker-job assignments
- determine optimal production schedules
- determine routes that will yield minimum shipping costs
- determine most profitable product mix

POM Applications

Aggregate planning
Production, Staffing
Distribution
Shipping
Inventory
Stock control, Supplier selection
Location
Plants or warehouses
Process management
Stock cutting
Scheduling
Shifts, Vehicles, Routing
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The Basic LP Question
A computer manufacturer makes two models of computers Type 1, and Type 2. The computers use many of the same components, made in the same factory by the same people and are stored in the same warehouse.

What Constrains What We Make?
- Materials
- Labor
- Time
- Cash
- Storage Space
- Shipping
- Customer Requirements
- Etc.

What Limits Us?

Components of Linear Programming
- **Objective** (e.g. maximize profits, minimize costs, etc.)
- **Decision variables** - those that can vary across a range of possibilities
- **Constraints** - limitations for the decision variables
- **Parameters** - the numerical values for the decision variables
- **Assumptions for an LP model**
  - linearity - the impact of the decision variables is linear in both constraints and objective function
  - divisibility - non-integer values for decision variables are OK
  - certainty - values of parameters are known and are constant
  - Non-negativity - decision variables >= 0
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Linear Programming Formulation

Maximize

x₁ = quantity of product 1 to produce
x₂ = quantity of product 2 to produce
x₃ = quantity of product 3 to produce

Subject To
Labor Constraint …………………………….. 2x₁ + 4x₂ + 8x₃ ≥ 50
Material Constraint………………………….. 7x₁ + 6x₂ + 5x₃ ≥ 100
Product 1 Constraint…………………………. x₁ ≤ 10
Non-negativity Constraint………………….. x₁, x₂, x₃ ≥ 0

Objective Function (maximize profit)………………… 5x₁ + 8x₂ + 4x₃

Relationships must be stated in terms of a relationship to a bound.
Suppose you have a ratio relationship as follows.

\[
\frac{x₁}{x₂} \leq \frac{3}{4}
\]

Mathematically simplifying this equation yields an equivalent equation. This simplified equation is more useful in the development of the linear programming optimal solution.

Constraint Equation Relationship Bound
4x₁ - 3x₂ ≤ 0

Graphical Linear Programming

When two decision variables (X₁ and X₂) are in the LP formulation, Graphical Linear Programming can be used to solve for the optimum values; however, when more than two decision variables are in the LP formulation, the graphical interpretation of the solution gets confusing and a computerized solution is required.

To understand the concepts of Linear Programming it is often educational to familiarize one’s self with the concepts of Graphical Linear Programming solutions. To this end we will consider the following example.

Graphical Linear Programming Solution
Example: A computer manufacturer makes two models of computers Type 1, and Type 2. The company resources available are also known. The marketing department indicates that it can sell whatever the company produces of either model. Find the quantity of Type 1 and Type 2 that will maximize company profits. The information available to the operations manager is summarized in the following table.

<table>
<thead>
<tr>
<th>Per Unit</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Amount Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profit</td>
<td>$60</td>
<td>$50</td>
<td></td>
</tr>
<tr>
<td>Assembly time (hr)</td>
<td>4</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>Inspection time (hr)</td>
<td>2</td>
<td>1</td>
<td>22</td>
</tr>
<tr>
<td>Storage space (cu ft)</td>
<td>3</td>
<td>3</td>
<td>39</td>
</tr>
</tbody>
</table>

First we must formulate the Linear Programming Problem.

**Objective Function (maximize profit)**

\[ 60x_1 + 50x_2 \]

**Subject To**

- **Assembly Time Constraint**
  \[ 4x_1 + 10x_2 \leq 100 \]
- **Inspection Time Constraint**
  \[ 2x_1 + x_2 \leq 22 \]
- **Storage Space Constraint**
  \[ 3x_1 + 3x_2 \leq 39 \]
- **Non-negativity Constraint**
  \[ x_1, x_2 \geq 0 \]

Next, we must plot each constraint (substituting the relationship with an equality sign). First plot the **Assembly Time Constraint**.

\[ 4x_1 + 10x_2 = 100 \]

The equation for this line can be plotted easily by solving the equation for the decision variable value when the other decision variable is 0. This gives the point where the line crosses each axis.

- \[ x_1 = \frac{100}{4} = 25 \]
- \[ x_2 = \frac{100}{10} = 10 \]
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Assembly Time Constraint

Feasible Region for Assembly Time – any point in this region will satisfy the Assembly Constraint Equation

Graphical Linear Programming - Example

Next plot the Inspection Time Constraint.

$2x_1 + 1x_2 = 22$

The equation for this line can be plotted easily by solving the equation for the decision variable value when the other decision variable is 0. This gives the point where the line crosses each axis.

$2x_1 + 1(0) = 22$

$x_1 = 11$

$2(0) + 1x_2 = 22$

$x_2 = 22$

Inspection Time Constraint

Feasible Region for Inspection Time - any point in this region will satisfy the Inspection Constraint Equation
Assembly & Inspection Time Constraints

Feasible Region for Assembly & Inspection Time – any point in this region will satisfy the both constraints.

Graphical Linear Programming - Example

Next plot the Storage Space Constraint.

3x₁ + 3x₂ = 39

The equation for this line can be plotted easily by solving the equation for the decision variable value when the other decision variable is 0. This gives the point where the line crosses each axis.

3x₁ + 3(0) = 39
x₁ = 13

3(0) + 3x₂ = 39
x₂ = 13

Storage Space Constraint

Feasible Region for Storage Space – any point in this region will satisfy the Storage Constraint Equation.
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Assembly, Inspection & Storage Constraints

Optimum Profit (maximum) is at one of the corner points of the feasible region. In this case, it is at the intersection ofStorage & Inspection.

By solving the simultaneous equations for Inspection and Storage, we obtain the optimum solution quantities for the decision variables.

\[\begin{align*}
2x_1 + x_2 &= 22 \\
3x_1 + 3x_2 &= 39
\end{align*}\]

Multiply top equation by -3

\[\begin{align*}
-6x_1 - 3x_2 &= -66 \\
3x_1 + 3x_2 &= 39
\end{align*}\]

Subtract the equations

\[\begin{align*}
-3x_1 &= -27 \\
x_1 &= 9
\end{align*}\]

Substitute in one of the equations

\[\begin{align*}
2(9) + x_2 &= 22 \\
x_2 &= 4
\end{align*}\]

Therefore, the optimum (maximum) profit is obtained when 9 Type 1 and 4 Type 2 computers are produced.

Graphical Linear Programming - Example

The amount of assembly time, inspection time, and storage space used at these optimum quantities are

<table>
<thead>
<tr>
<th>Category</th>
<th>Quantity</th>
<th>Slack</th>
<th>Binding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assembly</td>
<td>4(9) + 10(4) = 76</td>
<td>\leq 100</td>
<td></td>
</tr>
<tr>
<td>Inspection</td>
<td>2(9) + 1(4) = 22</td>
<td>\leq 22</td>
<td></td>
</tr>
<tr>
<td>Storage</td>
<td>3(9) + 3(4) = 39</td>
<td>\leq 39</td>
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Graphical Minimization Solutions

The previous example involved a Graphical Maximization Solution. Graphical Minimization Solutions are similar to that of maximization with the exception that one of the constraints must be = or >=. This causes the feasible solution space to be away from the origin. The other difference is that the optimal point is the one nearest the origin. We will not be doing any graphical minimization problems.

Other Linear Programming Terms

Redundant Constraint - one which does not form a unique boundary of the feasible solution space.

Feasible Solution Space - a polygon.

Optimal solution - a point or line segment on the feasible solution space. The optimal solution is always at one of the corner points of the polygon. In the case that the optimal solution is a line segment of the polygon, any point on the line segment will yield the same optimum solution.

Other Linear Programming Terms

Binding Constraint - one which forms the optimal corner point of the feasible solution space.

Surplus - when the values of decision variables are substituted into a >= constraint equation and the resulting value exceeds the right side of the equation.

Slack - when the values of decision variables are substituted into a <= constraint equation and the resulting value is less than the right side of the equation.
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Enter the data according to the linear programming formulation:
- Product Names
- Objective function coefficients
- Constraints
All of the constraint equations are plotted on the graph.

The optimal solution is automatically calculated showing the maximum value of the objective function and the quantities of each product that should be made to achieve it.

For this example: the maximum profit is achieved when 8 Type 1 and 4 Type 2's are manufactured.

Constraint utilization and slack/surplus is automatically calculated.
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Selecting the appropriate intersecting constraints shows the quantity points on the polygon where the objective function maximum is achieved.

EXCEL Solver LP - Example

Example: A computer manufacturer makes two models of computers Type 1, and Type 2. The company resources available are also known. The marketing department indicates that it can sell whatever the company produces of either model. Find the quantity of Type 1 and Type 2 that will maximize company profits. The information available to the operations manager is summarized in the following table.

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Step 3: Solve

Step 4: OK

The solution!

Quantities of each product that yield the maximum objective.

Objective function maximum.

Constraint utilization and Slack/Surplus.
The minimize LP looks exactly like the maximize LP. It functions exactly the same way and is used when the problem requires a optimum minimum solution.

**Linear Programming Formulation**

**Minimize**

\[ \text{Minimize cost} = 5x_1 + 8x_2 + 4x_3 \]

**Decision Variables**

- \( x_1 \) = quantity of product 1 to produce
- \( x_2 \) = quantity of product 2 to produce
- \( x_3 \) = quantity of product 3 to produce

**Subject To**

Constraints are stated in greater than or equal to terms rather than less than or equal to terms.

\[ 2x_1 + 4x_2 + 8x_3 \geq 250 \]
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EXCEL Solver LP Templates
Read and understand all material in the chapter.
Discussion and Review Questions
Recreate and understand all classroom examples
Exercises on chapter web page

Appendix: EXCEL Solver Details
If you ever have to do your own solver, the following slides detail the steps in Excel you should follow.